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Binary option pricing using fuzzy numbers

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ABSTRACT

A binary option is a type of option where the payout is either fixed after the underlying stock exceeds the predetermined threshold (or strike price) or is nothing at all. Traditional option pricing models determine the option's expected return without taking into account the uncertainty associated with the underlying asset price at maturity. Fuzzy set theory can be used to explicitly account for such uncertainty. Here we use fuzzy set theory to price binary options. Specifically, we study binary options by fuzzifying the maturity value of the stock price using trapezoidal, parabolic and adaptive fuzzy numbers.

1. Introduction

A standard option is a contract that gives the holder the right to buy or sell an underlying asset at a specified price on a specified date. The payoff depends on the underlying asset price. The call option gives the holder the right to buy an underlying asset at a strike price; the strike price is termed a specified price or exercise price. Therefore the higher the underlying asset price, the more valuable the call option. If the underlying asset price falls below the strike price, the holder would not exercise the option. The binary option is an exotic call option with discontinuous payoffs. The option pays off a fixed, predetermined amount if the underlying asset price is beyond the strike price on its expiration date. There are two kinds of binary options: asset-or-nothing call options and cash-or-nothing call options. For the first type, the option pays off nothing if the underlying asset price ends up below the strike price. For the second type, the option pays off nothing if the underlying asset price ends up below the strike price and pays a fixed amount if it ends up above the strike price. Note that for the binary option the underlying asset is the stock and the underlying asset price is termed the stock price. The traditional binary option pricing model is shown in Section 1.1. As can be seen, the model does not take into account the uncertainty associated with the underlying asset price at maturity, S_T. Fuzzy set theory can be used to explicitly account for such uncertainty [1]. We use fuzzy numbers to provide an alternative model to option pricing. Carlsson and Fuller [2] were the first to study the fuzzy real options. Thavaneswaran et al. [3] demonstrated the superiority of the fuzzy forecasts and then derived the membership function for the European call price by fuzzifying the interest rate, volatility and the initial value of the stock price. Other studies such as Guerra et al. [4] and Chrysafis and Papadopoulos [5] have used fuzzy numbers in option pricing; however binary options have been little explored. Zmeskal [6] proposed a fuzzy binomial American real option model. In this paper, we study the asset-or-nothing European option by fuzzifying the maturity value of the stock price. In Section 1.2 we introduce the basics of fuzzy numbers. In Section 2 we derive the asset-or-nothing fuzzy European option pricing model.

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1.1. The asset-or-nothing call option

If the stock price never hits the strike price a at expiration, then the option is worthless; thus at or below a, the option value is zero. If S_T surpasses the price a, we let the final payment of the option be S_T (the stock price at maturity). If $C(S_T)$ is the value of the asset-or-nothing call option on its expiration date, then the final boundary condition is

$$C(S_T) = \begin{cases} S_T & S_T > a \\ 0 & S_T \leq a. \end{cases}$$

With the assumption that the expected return is the risk-free interest rate, we get

$$C = e^{-r(T-t)} EC(S_T) \quad \text{such that } 0 \leq t \leq T. \tag{1.1}$$

1.2. Fuzzy numbers

We follow the notation and concepts introduced in [2,7].

Definition 1.1. A fuzzy set A in $X \subset \Re$, where $\Re$ is the set of real numbers, is a set of ordered pairs $A = \{(x, \ \mu(x)) : x \in X\}$, where $\mu(x)$ is the membership function or grade of membership, or degree of compatibility or degree of truth of $x \in X$ which maps $x \in X$ on the real interval [0, 1].

Definition 1.2. A fuzzy set A in $\Re^n$ is said to be a convex fuzzy set if its γ-level sets $A(\gamma)$ are (crisp) convex sets for all $\gamma \in [0, 1]$. Alternatively, a fuzzy set A in $\Re^n$ is a convex fuzzy set if and only if for all $x_1, x_2 \in \Re^n$ and $0 \leq \lambda \leq 1$,

$$\mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \mathrm{Min}(\mu_A(x_1), \mu_A(x_2)).$$

Definition 1.3. A fuzzy number $\tilde{A} \in \mathcal{F}$ is called a trapezoidal fuzzy number (Tr.F.N.) with core $[a, b]$, left width α and right width β if its membership function has the following form:

$$g(x) = \begin{cases} 1 - \dfrac{a-x}{\gamma} & \text{if } a - \gamma \leq x \leq \gamma \\ 1 & \text{if } a \leq x \leq b \\ 1 - \dfrac{x-b}{\beta} & \text{if } a \leq x \leq b + \beta \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

and we use the notation $\tilde{A} = (a, b, \gamma, \beta)$. It can easily be shown that

$$A(\alpha) = [a_1(\alpha), a_2(\alpha)] = [a - (1 - \alpha)\gamma, b + (1 - \alpha)\beta] \quad \forall \alpha \in \ [0, 1]. \tag{1.3}$$

The support of $\tilde{A}$ is $(a-\gamma, b-\beta)$. Moreover, for any fuzzy number $\tilde{A}$ and a positive real number C, the following relationship holds:

$$\tilde{A} \leq C \iff \int_0^1 (a_1(\alpha) + a_2(\alpha))\alpha d\alpha \leq C. \tag{1.4}$$

Definition 1.4. Let $\Re$ be the set of all real numbers. A fuzzy number $G(x)$, $x \in \Re$ is of the form

$$G(x) = \begin{cases} g(x) & \text{when } x \in [a, b) \\ 1 & \text{when } x \in [b, c) \\ h(x) & \text{when } x \in [c, d) \\ 0 & \text{otherwise} \end{cases} \tag{1.5}$$

where g is a real valued, increasing and right continuous function, h is a real valued, decreasing and left continuous function, and a, b, c, d are real numbers such that $a < b < c < d$. A fuzzy number A with shape functions g and h defined by

$$g(x) = \left(\frac{x-a}{b-a}\right)^m \tag{1.6}$$

$$h(x) = \left(\frac{d-x}{d-c}\right)^n \tag{1.7}$$

respectively, where m or $n > 0$, will be denoted by $A = [a, b, c, d]_{m,n}$. If $m = 1$ and $n = 1$, we simply write $A = [a, b, c, d]$, which is known as a trapezoidal fuzzy number. If $m \neq 1$ or $n \neq 1$, a fuzzy number $A^* = [a, b, c, d]_{m,n}$ is a modification of a trapezoidal fuzzy number $A = [a, b, c, d]$. If $m > 1$ and $n > 1$, then A^* is a concentration of A. Concentration of A by m and $n = 2$ is often interpreted as the linguistic hedge "very". If $0 < (m \text{ or } n) < 1$, then A^* is a dilation of A. Dilation of A by *m and* $n = 0.5$ is often interpreted as the linguistic hedge "more or less". Each fuzzy number A described by (1.6) and (1.7) has the following α-level sets, $A(\alpha) = [a(\alpha), b(\alpha)], a(\alpha), \ b(\alpha) \in \Re, \ \alpha \in [0, 1]$ and

$$A(\alpha) = [g^{-1}(\alpha), h^{-1}(\alpha)], \qquad A_1 = [b, c], \qquad A_0 = [a, d].$$

If $A = [a, b, c, d]_{m,n}$ then, for all $\alpha \in [0, 1]$,

$$A(\alpha) = [a + \alpha^{\frac{1}{m}}(b - a), d - \alpha^{\frac{1}{n}}(d - c)]. \tag{1.8}$$

2. The asset-or-nothing fuzzy pricing model

2.1. General terminal-value claims

The method of pricing the European call option can be used to find the price of any claim in the Black–Scholes model:

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

that is

$$S_t = S_0 e^{\sigma W_t + (\mu - \sigma^2/2)t},$$

where μ and σ represent the expected return and volatility per unit time, respectively, and $\{W_t\}$ is a Wiener process. The price at time 0 of a claim paying C at time T is $\mathbb{E}_Q[e^{-rT}C]$, where taking expectations with the martingale probability Q gives the same value as taking expectations with the original probabilities with the assumption that $\mu = r$; the price at time t will be $\mathbb{E}_Q[e^{-r(T-t)}C|\mathcal{F}_t]$. Here C may be any $\mathcal{F}_T$ random variable with $\mathbb{E}[C^2] < \infty$.The following theorem gives the time t price of a general terminal-value claim $C = f(S_T)$.

Theorem 2.1. (a) *The time t price of the terminal-value claim* $C = (S_T)^\nu$ *for some real number* ν *is* $(S_t)^\nu e^{-(1-\nu)(r+\nu\sigma^2/2)(T-t)}$,
(b) *The time t price of the asset-or-nothing claim* $C = (S_T)^\nu I\{a(\alpha) \leq S_T \leq b(\alpha)\}$ *is*

$$(S_t)^\nu e^{((\nu-1)r + \frac{1}{2}\nu(\nu-1)\sigma^2)(T-t)}[\Phi\,(d_\nu(b)) - \Phi(d_\nu(a))], \tag{2.1}$$

where

$$d_\nu(u) = \frac{\ln\left(\frac{u}{S_t}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} - \nu\sigma\sqrt{T-t}, \quad \textit{and}$$

$a(\alpha)$ *and* $b(\alpha)$ *are the upper and lower* α*-cuts, respectively.*
(c) *For any twice-differentiable function* $f : (0, \infty) \to \mathbb{R}$*, the time t price of the terminal-value claim* $C = f(S_T)I\{a(\alpha) \leq S_T \leq b(\alpha)\}$ *is given by*

$$p(x, t) = e^{-r(T-t)}\mathbb{E}[f(xe^X)I\{a(\alpha) \leq S_T \leq b(\alpha)\}]$$

where $X = \sigma Z\sqrt{T-t} + (r - \sigma^{2/2})(T-t)$,

Proof. The time t price is given by $\mathbb{E}_Q[e^{-r(T-t)}(S_T)^\nu|\mathcal{F}_t] = S_t^\nu e^{-(1-\nu)(r+\nu\sigma^{2/2})(T-t)}$ and hence (a) follows. For part (b),

$$\begin{aligned}
&\mathbb{E}_Q[e^{-r(T-t)}(S_T)^\nu I_{\{a(\alpha)\leq S_T\leq b(\alpha)\}}|\mathcal{F}_t] \\
&\quad = \mathbb{E}[e^{-r(T-t)}(S_t)^\nu e^{\nu(r-\sigma^2/2)(T-t)+\nu\sigma(W_T-W_t)} I_{\{a(\alpha)\leq S_t e^{(r-\sigma^2/2)(T-t)+\sigma(W_T-W_t)}\leq b(\alpha)\}}|\mathcal{F}_t] \\
&\quad = (S_t)^\nu e^{((\nu-1)r-\frac{\nu\sigma^2}{2})(T-t)}\mathbb{E}[e^{\nu\sigma(W_T-W_t)} I_{\{\ln(\frac{a(\alpha)}{S_t})-(r-\frac{\sigma^2}{2})(T-t)\leq\sigma(W_T-W_t)\leq\ln(\frac{b(\alpha)}{S_t})-(r-\frac{\sigma^2}{2})(T-t)\}}|\mathcal{F}_t] \\
&\quad = (S_t)^\nu e^{((\nu-1)r-\frac{\nu\sigma^2}{2})(T-t)}\mathbb{E}[e^{\nu X} I_{\{\ln(\frac{a(\alpha)}{S_t})-(r-\frac{\sigma^2}{2})(T-t)\leq X\leq\ln(\frac{b(\alpha)}{S_t})-(r-\frac{\sigma^2}{2})(T-t)\}}] \\
&\quad = ((S_t)^\nu e^{((\nu-1)r-\frac{\nu\sigma^2}{2})(T-t)} e^{\frac{1}{2}\nu^2\sigma^2(T-t)}) \\
&\qquad \times \mathbb{P}\left(\ln\left(\frac{a(\alpha)}{S_t}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t) \leq X + \nu\sigma^2(T-t) \leq \ln\left(\frac{b(\alpha)}{S_t}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)\right) \\
&\quad = ((S_t)^\nu e^{((\nu-1)r+\frac{1}{2}\nu(\nu-1))(T-t)}) \\
&\qquad \times \mathbb{P}\left(\frac{\ln\left(\frac{a(\alpha)}{S_t}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} - \nu\sigma\sqrt{T-t} \leq Z \leq \frac{\ln\left(\frac{b(\alpha)}{S_t}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} - \nu\sigma\sqrt{T-t}\right) \\
&\quad = (S_t)^\nu e^{\left((\nu-1)r+\frac{1}{2}\nu(\nu-1)\sigma^2\right)(T-t)}[\Phi(d_\nu(b(\alpha))) - \Phi(d_\nu(a(\alpha)))],
\end{aligned}$$

where

$$d_\nu(u) = \frac{\ln\left(\frac{u}{S_t}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} - \nu\sigma\sqrt{T-t},$$

and hence (b) follows.

Table 1 presents a numerical example of the price of the asset-or-nothing terminal-value claim. We use the S&P 100 data in [8] for the period 1991–2000, for three different expiration dates (T) and five α values.

The proof of (c) is somewhat similar to the proof given for crisp valued S_T in [9]:

$$p(x,t) = e^{-r(T-t)}\mathbb{E}_Q[f(S_T)I\{a(\alpha) \le S_T \le b(\alpha)\}|\mathcal{F}_t] \tag{2.2}$$

$$= e^{-r(T-t)}\mathbb{E}[f(xe^X)I\{a(\alpha) \le S_T \le b(\alpha)\}], \tag{2.3}$$

where $X = \sigma Z\sqrt{T-t+(r-\sigma^2/2)(T-t)}$. Differentiating with respect to x, we obtain

$$\frac{\partial p}{\partial x} = e^{-r(T-t)}\mathbb{E}[e^X f'(xe^X)I\{a(\alpha) \le S_T \le b(\alpha)\}] \tag{2.4}$$

$$= \mathbb{E}[f'(xe^X)I\{a(\alpha) \le S_T \le b(\alpha)\}]. \tag{2.5}$$

Similarly,

$$\frac{\partial^2 p}{\partial x^2} = \mathbb{E}[e^X f''(xe^X)I\{a(\alpha) \le S_T \le b(\alpha)\}].$$

Differentiating (2.3) with respect to t,

$$\frac{\partial p}{\partial t} = rp - xe^{-r(T-t)}\mathbb{E}\left[\frac{\sigma Z}{2\sqrt{T-t}} + \left(r - \frac{\sigma^2}{2}\right)e^{X-\sigma^2(T-t)}f'(xe^{X-\sigma^2(T-t)})I\{a(\alpha) \le S_T \le b(\alpha)\}\right] \tag{2.6}$$

$$= rp - x\mathbb{E}\left[\left(\frac{\sigma Z}{2\sqrt{T-t}} + r\right)f'(xe^X)I\{a(\alpha) \le S_T \le b(\alpha)\}\right] \tag{2.7}$$

$$= rp - \frac{\sigma x}{2\sqrt{T-t}}\mathbb{E}[Xf'(xe^X)I\{a(\alpha) \le S_T \le b(\alpha)\}] - rx\frac{\partial p}{\partial x}. \tag{2.8}$$

Since

$$\begin{aligned}\mathbb{E}[Xf'(xe^X)I\{a(\alpha) \le S_T \le b(\alpha)\}] &= \sigma x\sqrt{T-t}\,\mathbb{E}[e^X f''(xe^X)I\{a(\alpha) \le S_T \le b(\alpha)\}] \\ &= \sigma x\sqrt{T-t}\frac{\partial^2 p}{\partial x^2},\end{aligned}$$

we have

$$\frac{\partial p}{\partial t} = rp - \frac{1}{2}\sigma^2 x^2\frac{\partial^2 p}{\partial x^2} - rx\frac{\partial p}{\partial x} \tag{2.9}$$

$$= r\left(p - x\frac{\partial p}{\partial x}\right) - \frac{1}{2}\sigma^2 x^2\frac{\partial^2 p}{\partial x^2}. \quad \square \tag{2.10}$$

Next, we present four examples of call option prices using fuzzy numbers for S_T using membership functions leading to trapezoidal, adaptive, parabolic, and elliptic fuzzy numbers.

Example 1. For the asset-or-nothing option stated above, the fuzzy concept is taken into account in a model. Define a fuzzy set as a trapezoidal fuzzy number with core $[S_a, S_b]$, left width α and right width β. Consider the membership function related to the asset price which follows the trapezoid function. If they introduce the fuzzy concept into a binary function, investors may have more opportunities to think about their decisions as regards some aspect such as risk.

$$g(S(T)) = \begin{cases} 1 - \left(\frac{S_a - S(T)}{\gamma}\right) & \text{if } S_a - \gamma \le S(T) \le S_a \\ 1 & \text{if } S_a \le S(T) \le S_b \\ 1 - \left(\frac{S(T) - S_b}{\beta}\right) & \text{if } S_b \le S(T) \le S_b + \beta \\ 0 & \text{otherwise} \end{cases} \tag{2.11}$$

where $\gamma \ge 0$, $\beta \ge 0$.

Table 1
Price of the asset-or-nothing terminal-value claim C for S&P 100 data, 1991–2000 ($\sigma^2 = 0.16$, $\nu = 2$, $r = 3\%$).

T	$\bar{S}_T$	α	$Z_{\alpha/2}$	$a(\alpha)$	$b(\alpha)$	Terminal-value claim	
						Lower	Upper
24	425.53	0.025	2.2414	425.3258	425.7242	81.7872	81.9404
		0.050	1.9600	425.3508	425.6992	71.5261	71.6433
		0.100	1.6449	425.3788	425.6712	60.0345	60.1170
		0.500	0.6745	425.4651	425.5849	24.6278	24.6416
		0.900	0.1257	425.5138	425.5362	4.5893	4.5898
87	425.67	0.025	2.2414	425.5724	425.7576	16.4399	16.4542
		0.050	1.9600	425.5840	425.7460	14.3764	14.3874
		0.100	1.6449	425.5970	425.7330	12.0658	12.0735
		0.500	0.6745	425.6371	425.6929	4.9487	4.9500
		0.900	0.1257	425.6598	425.6702	0.9221	0.9221
115	425.55	0.025	2.2414	425.3557	425.7368	26.9479	26.9962
		0.050	1.9600	425.3796	425.7129	23.5668	23.6038
		0.100	1.6449	425.4064	425.6861	19.7804	19.8064
		0.500	0.6745	425.4889	425.6036	8.1143	8.1187
		0.900	0.1257	425.5356	425.5569	1.5121	1.5122

The most possible values of the underlying asset price at the maturity date lie in the interval $[S_a, S_b]$, and $S_b + \beta$ is the upward potential and $S_a+\alpha$ is the downward potential for the values of the underlying asset price. For fixed parameter values of α, β, S_a, and S_b there are many ways to consider. For example, when the investor cannot predict how the underlying asset price changes at the maturity date, in other words, on becoming confident that the asset price has fluctuated greatly, the investor will take the range of sufficiently large width that the premium values become high. On the other hand, when much fluctuation is not observed the width will become small, so S_a becomes equal to S_b, resulting in triangular fuzzy numbers. For each of the three sets the corresponding payoff is obtained by multiplying its grade of membership function, ϕ_S.

In this case, and the underlying asset $S(T)$ moves between $S_a - \gamma$ and $S_b + \beta$. Then, the present value of option may be computed as a difference between the present value of $S(T)$ which exceeds $S_a - \gamma$ and that of $S(T)$ which is above $S_b + \beta$.

$$\text{payoff} = g(S(T)) \times S(T) = \begin{cases} S(T) - \left(\dfrac{S_a S(T) - S(T)^2}{\gamma}\right) & \text{if } S_a - \gamma \le S(T) \le S_a \\ S(T) & \text{if } S_a \le S(T) \le S_b \\ S(T) - \left(\dfrac{S(T)^2 - S_b S(T)}{\beta}\right) & \text{if } S_b \le S(T) \le S_b + \beta \\ 0 & \text{otherwise.} \end{cases}$$

Then, the values of the asset-or-nothing option with fuzzy nature are as follows:

$$C = C_1 - C_2 - C_3$$

where

$$\begin{aligned} C_1 &= e^{-r(T-t)}\mathbb{E}_Q[S(T)], \quad S_a - \gamma < S(T) < S_b + \beta \\ C_2 &= e^{-r(T-t)}\left(\frac{S_a\mathbb{E}_Q[S(T)] - \mathbb{E}_Q[S(T)^2]}{\gamma}\right), \quad S_a - \alpha < S(T) < S_a \\ C_3 &= e^{-r(T-t)}\left(\frac{S_a\mathbb{E}_Q[S(T)^2] - S_b\mathbb{E}_Q[S(T)]}{\beta}\right), \quad S_b < S(T) < S_b + \beta \end{aligned}$$

with appropriate boundary conditions, where $\mathbb{E}_Q$ denotes the conditional expectation with respect to risk-neutral probability.

$$\begin{aligned} C_1 &= e^{-r(T-t)}\mathbb{E}_Q[S(T)], \quad S_a - \gamma < S(T) < S_b + \beta \\ &= e^{-r(T-t)}\mathbb{E}_Q[S(T)] \\ &= e^{-r(T-t)}S_t[\phi(d(S_b + \beta)) - \phi(d(S_a - \gamma))] \end{aligned}$$

and

$$\begin{aligned} C_2 &= e^{-r(T-t)}\left(\frac{S_a\mathbb{E}_Q[S(T)|\mathcal{F}_t]}{\gamma} - \frac{\mathbb{E}_Q[S(T)^2|\mathcal{F}_t]}{\gamma}\right), \quad S_a - \gamma < S(T) < S_a \\ &= e^{-r(T-t)}\left(\frac{S_a\mathbb{E}_Q[S(T)|\mathcal{F}_t]}{\gamma} - \frac{\mathbb{E}_Q[S(T)^2|\mathcal{F}_t]}{\gamma}\right) = e^{-r(T-t)}\left(\frac{S_a S(T)}{\gamma} - \frac{S^2(T)}{\gamma}\right) \end{aligned}$$

$$= e^{-r(T-t)}\left(\frac{S_a S_t[\phi(d(S_a)) - \phi(d(\, S_a - \gamma))]}{\gamma} - \frac{(S_t)^2 e^{\left((2-1)r+\frac{1}{2}2(2-1)\sigma^2\right)(T-t)}[\phi(d_2(S_a)) - \phi(d_2(\, S_a - \gamma))]}{\gamma}\right)$$

$$= e^{-r(T-t)}\left(\frac{S_a S_t[\phi(d(S_a)) - \phi(d(\, S_a - \alpha))]}{\gamma} - \frac{S_t^2 e^{(r+\sigma^2)(T-t)}[\phi(d_2(S_a)) - \phi(d_2(\, S_a - \gamma))]}{\gamma}\right)$$

and

$$C_3 = e^{-r(T-t)}\left(\frac{\mathbb{E}_Q[S(T)^2|\mathcal{F}_t]}{\beta} - \frac{S_b \mathbb{E}_Q[S(T)|\mathcal{F}_t]}{\beta}\right)$$
$$= e^{-r(T-t)}\left(\frac{S_t^2 e^{(r+\sigma^2)(T-t)}[\phi(d_2(S_b+\beta)) - \phi(d_2(S_b))]}{\beta} - \frac{S_b S_t[\phi(d(S_b+\beta)) - \phi(d(S_b))]}{\beta}\right).$$

Example 2. When we model the terminal value S_T by an adaptive fuzzy number having membership function of the form

$$g(S(T)) = \begin{cases} 1 - \left(\dfrac{S_a - S(T)}{\gamma}\right)^n & \text{if } S_a - \gamma \le S(T) \le S_a \\ 1 & \text{if } S_a \le S(T) \le S_b \\ 1 - \left(\dfrac{S(T) - S_b}{\beta}\right)^n & \text{if } S_b \le S(T) \le S_b + \beta \\ 0 & \text{otherwise} \end{cases} \tag{2.12}$$

and if the payoff is given by

$$g(S(T)) \times S(T) = \begin{cases} S(T) - \left(\dfrac{S_a - S(T)}{\gamma}\right)^n S(T) & \text{if } S_a - \gamma \le S(T) \le S_a \\ S(T) & \text{if } S_a \le S(T) \le S_b \\ S(T) - \left(\dfrac{S(T) - S_b}{\beta}\right)^n S(T) & \text{if } S_b \le S(T) \le S_b + \beta \\ 0 & \text{otherwise,} \end{cases} \tag{2.13}$$

then the time t call price is given by $C = C_1 - C_2 - C_3$, where

$$C_1 = e^{-r(T-t)}\mathbb{E}_Q[S(T)|\mathcal{F}_t], \quad S_a - \gamma < S(T) < S_b + \beta$$
$$= S_t[\phi(d(S_b+\beta)) - \phi(d(S_a-\gamma))]$$

and

$$C_2 = e^{-r(T-t)}\mathbb{E}_Q\left([S(T)]\left(\frac{S_a}{\gamma} - \frac{S(T)}{\gamma}\right)^n \middle| \mathcal{F}_t\right) = e^{-r(T-t)}\left(\frac{1}{\gamma}\right)^n \mathbb{E}_Q\left(S(T)\left(\frac{S_a - S(T)}{\gamma}\right)^n \middle| \mathcal{F}_t\right)$$
$$= e^{-r(T-t)}\left(\frac{1}{\gamma}\right)^n \mathbb{E}_Q[S(T)(S_a - S(T))^n|\mathcal{F}_t] = e^{-r(T-t)}\left(\frac{1}{\gamma}\right)^n \mathbb{E}_Q\left(S(T)\sum_{k=0}^{n}\binom{n}{k} S_a^{n-k}(-1)^k S^k(T)|\mathcal{F}_t\right).$$

It follows from Theorem 2.1 that

$$C_2 = e^{-r(T-t)}\left(\frac{1}{\gamma}\right)^n \left(\begin{array}{c}\displaystyle\sum_{k=0}^{n}\binom{n}{k} S_a^{n-k}(-1)^k(S_t)^{k+1}e^{\left(kr+\frac{1}{2}(k+1)k\sigma^2\right)(T-t)} \\ [\phi(d_{k+1}(S_a)) - \phi(d_{k+1}(S_a-\gamma))]\end{array}\right).$$

Similarly,

$$C_3 = e^{-r(T-t)}\mathbb{E}_Q\left(S(T)\left(\frac{S(T)-S_b}{\beta}\right)^n\right) = e^{-r(T-t)}\left(\frac{1}{\beta}\right)^n \mathbb{E}_Q(S(T)(S(T)-S_b)^n)$$
$$= e^{-r(T-t)}\left(\frac{1}{\beta}\right)^n\left(\sum_{k=0}^{n}\binom{n}{k}\left(\left(\begin{array}{c}(S_t)^{n-k+1}e^{\frac{1}{2}(-n+k)(-2r-\sigma^2 n+\sigma^2 k-\sigma^2)(T-t)} \\ [\phi(d_{n-k+1}(S_b+\beta)) - \phi(d_{n-k+1}(S_b))]\end{array}\right)\right)(-1)^k S_b^k\right).$$

Example 3. If we model the terminal value of the stock price $S(T)$ as a parabolic fuzzy number with membership function

$$g(S(T)) = \frac{2(a - S(T))(S(T) - 2a)}{3a}, \quad a \le S(T) \le 2a$$

then it follows from Theorem 2.1 that the time t price of the claim $S(T)g(S(T)) = C_1 - C_2 - C_3$, where

$$C_1 = 2(S_t^2 e^{(r+\sigma^2)(T-t)}[\phi(d_2(2a)) - \phi(d_2(a))]),$$
$$C_2 = \frac{4a_1}{3}(S_t[\phi(d_1(2a)) - \phi(d_1(a))]),$$
$$C_3 = \frac{2[S_t^3 e^{(2r+3\sigma^2)(T-t)}[\phi(d_3(2a)) - \phi(d_3(a))]]}{3a},$$

and

$$d_\nu(u) = \frac{\ln\left(\frac{u}{S_t}\right) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} - \nu\sigma\sqrt{T-t}.$$

Example 4. If we model the terminal value of the stock price $S(T)$ as an elliptic fuzzy number with membership function

$$g(S(T)) = \frac{4}{b^2}(a - S(T))(S(T) - (a+b)), \quad a \le S(T) \le a+b$$

for positive a, b, then the time t price of the claim $S(T)g(S(T)) = C_1 - C_2 - C_3$, where

$$C_1 = \left(\frac{8a}{b^2} + \frac{4}{b}\right) S_t^2 e^{(r+\sigma^2)(T-t)}[\phi(d_2(a+b)) - \phi(d_2(a))],$$
$$C_2 = \left(\frac{4}{b^2}a^2 + 4\frac{4}{b}a\right) S_t[\phi(d_1(a+b)) - \phi(d_1(a))],$$
$$C_3 = \frac{4}{b^2} S_t^3 e^{(2r+3\sigma^2)(T-t)}[\phi(d_3(a+b)) - \phi(d_3(a))],$$

and

$$d_\nu(u) = \frac{\ln\left(\frac{u}{S_t}\right) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} - \nu\sigma\sqrt{T-t}.$$

3. Conclusions

Motivated by the findings of Thavaneswaran et al. [3,10] showing the superiority of fuzzy forecasts relative to minimum square error forecasts and the application of fuzzy numbers to option pricing, we model the general terminal value of the stock price S_T as a fuzzy number. We then derive the asset-or-nothing call price for the fuzzified S_T using the Black–Scholes option pricing formula and we present a numerical example using S&P 100 index data. Previous research has provided call option pricing results using fuzzy volatility; however, the terminal value of fuzzy stock prices has not been explored. We also present four examples of call option prices for fuzzy values of the stock price at maturity. In the examples, we derive the expressions for the time t call price when the terminal value of the stock price is modeled using membership functions leading to trapezoidal, adaptive, parabolic, and elliptic fuzzy numbers.

In our fuzzy option pricing model we assume constant volatility. Previous studies have shown that if we relax this assumption we may explain volatility better, but no research has been performed for binary options in a fuzzy environment. Future research could extend our model by incorporating time-varying volatility.

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References

[1] W. Xu, W. Xu, H. Li, W. Zhang, A study of Greek letters of currency option under uncertainty environments, Mathematical and Computer Modelling 51 (5–6) (2010) 670–681.
[2] C. Carlsson, R. Fuller, On possibilistic mean value and variance of fuzzy numbers, Fuzzy Sets and Systems 122 (2001) 315–326.
[3] A. Thavaneswaran, S.S. Appadoo, A. Paseka, Weighted possibilistic moments of fuzzy numbers with applications to GARCH modeling and option pricing, Mathematical and Computer Modelling 49 (2009) 352–368.

[4] M.L. Guerra, L. Sorini, L. Stefanini, Option price sensitivities through fuzzy numbers, Computers and Mathematics with Applications 61 (3) (2011) 515–526.
[5] K. Chrysafis, B. Papadopoulos, On theoretical pricing of options with fuzzy estimators, Journal of Computational and Applied Mathematics 223 (2) (2009) 552–566.
[6] Z. Zmeskal, Generalized soft binomial American real option model (fuzzy–stochastic approach), European Journal of Operational Research 207 (2010) 1096–1103.
[7] H.J. Zimmermann, Fuzzy Set Theory and its Applications, fourth ed., Kluwer Academic Publishers, Nowell, 2001.
[8] H. Gong, A. Thavaneswaran, J. Singh, A Black–Scholes model with GARCH volatility, Mathematical Scientist 35 (2010) 37–42.
[9] D. Kennedy, Stochastic financial models, in: Taylor and Francis Group, Cambridge, 2010.
[10] A. Thavaneswaran, J. Singh, S.S. Appadoo, Option pricing for some stochastic volatility models, The Journal of Risk Finance 7 (4) (2006) 425–445.